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Nonlinear Fluctuations, Separation of Procedures, and Linearization of Processes

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A concept of separation of procedures is introduced to study cooperative phenomena theoretically. Some typical important examples of this concept are presented to clarify its usefulness; Kubo's stochastic Liouville equation, some generalized diffusionlike equations, van Kampen's expansion, Kubo's extensivity, Prigogine's entropy production, the scaling theory of transient phenomena based on the Lie algebra, and Suzuki's CAM theory of cooperative phenomena are discussed from the new viewpoint of separation of procedures.

KEY WORDS: Separation of procedures; nonlinear fluctuation; stochastic Liouville equation; diffusion equation; Fokker–Planck equation; Ω-expansion; extensivity; entropy production; scaling theory; coherent-anomaly method.

1. INTRODUCTION

Many interesting phenomena, such as phase transitions, occur due to the nonlinearity of the relevant system, as is well known.[1–5] However, what is the nonlinearity? It may depend on the representation of each phenomenon. For example, a nonlinear Langevin equation[6–8] of the form

$$\frac{d}{dt}x = \alpha(x) + \eta(t), \qquad \langle \eta(t)\,\eta(t') \rangle = 2\varepsilon\delta(t-t') \tag{1}$$

with a nonlinear function $\alpha(x)$ and a Gaussian white noise $\eta(t)$ is equivalently described by the following Fokker–Planck equation[6–8]:

$$\frac{\partial}{\partial t}P(x,t) = \left\{ -\frac{\partial}{\partial x}\alpha(x) + \varepsilon\frac{\partial^2}{\partial x^2} \right\} P(x,t) \tag{2}$$

This paper is dedicated to Ilya Prigogine on the occasion of his 70th birthday.

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This is linear with respect to the distribution function $P(x, t)$. Similarly, the deterministic nonlinear differential equation $dx/dt = \alpha(x)$ can be transformed into an infinite number of linear differential equations by the Poincaré–Carleman method.[9] This corresponds to the special case $\varepsilon = 0$ in (1) and (2). In general, nonlinear equations can be expressed by linear ones in a wider representation space. A similar situation appears in the classification of Markovian and non-Markovian processes.

Next we classify nonlinearity into two categories, namely *weak nonlinearity*, which can be transformed into a linear case by a monotonic transformation of variables, and *strong nonlinearity*, which *cannot* be transformed into a linear case by any monotonic transformation.

It will be instructive to discuss here a well-known example, namely the van der Waals equation. The equation of state for ideal gas is given by $PV = Nk_B T$ (where P, V, N, and T denote the pressure, molar volume, Avogadro number, and temperature, respectively). This is a linear equation with respect to the pressure and number density $\rho = N/V$. Maxwell introduced a transformation $V \to V - b$ to obtain his equation of state $P(V-b) = Nk_B T$ in order to take into account the scale of molecules or their hard-core effect. His equation has only weak nonlinearity and consequently no phase transition appears in his equation. In 1873, van der Waals proposed the following celebrated equation of state:

$$\left(P + \frac{a}{V^2}\right)(V-b) = Nk_B T \tag{3}$$

This is rewritten as

$$P = \frac{Nk_B T}{V-b} - \frac{a}{V^2} \equiv f(V) - g(V) \tag{4}$$

Clearly, this shows strong nonlinearity and it can describe the gas–liquid phase transition qualitatively, as is well known. It should be noted here that the above two terms $f(V)$ and $g(V)$ in (4) give remarkable effects in different regions of the volume V, and consequently that strong nonlinearity appears in (3) or (4).

Strong nonlinearity can often appear as a competition between two kinds of weak nonlinearity that each shows an effect in different regions of the relevant variable. This remark will be useful in explaining or discovering interesting nonlinear phenomena. For example, the so-called Alder transition[10] of hard-core systems can be interpreted as a typical example of the above general situation, namely the boundary condition that the relevant system is confined to a finite volume (even a periodic boundary condition) plays the role of an attractive interaction for long distances.

As was emphasized by Prigogine,[1-4] fluctuations play an essential role in the formation of macroscopic order.[6-8] In particular, nonlinear fluctuations are substantial in the modern analysis of phase transitions and critical phenomena.[5] Thus, the entanglement of nonlinearity and fluctuations is one of the most important subjects in statistical physics, as will be discussed later in detail.

One of the main purposes in the present paper is to introduce the concept[2] of "separation of procedures" or "linearization of processes" to study nonlinear phenomena theoretically, as will be explained generally in Section 2. Many applications of this concept to physical problems will be given in the succeeding sections.

2. SEPARATION OF PROCEDURES AND LINEARIZATION OF PROCESSES

What is theoretical physics? It aims at understanding the laws of nature and its mechanisms in order to control various kinds of interesting phenomena. The simplest way to understand them is to find some linear relation between the cause and result. When fundamental equations to describe phenomena are nonlinear and complicated, we have to solve them analytically or numerically. What do we mean by solving nonlinear equations analytically? This means, I believe, nothing but transforming original nonlinear equations into linear ones by introducing some appropriate nonlinear transformation of variables. Namely, it is the linearization of processes. This strategy is very useful in studying nonlinear systems theoretically, as will be exemplified later.

More conceptually or from a methodological point of view, it is convenient to introduce the concept of *separation of procedures, by which we mean separating our procedures to solve the relevant nonlinear problems into several steps, each of which is described by an explicit tractable analytic expression.*

More explicitly, for example, we consider some exponential procedure $\exp(A+B)$ for noncommutable operators A and B. If the exponential operator $\exp(A+B)$ is decomposed into the form

$$\exp(A+B) = f(A)\, h(B) \tag{5}$$

then we have separation of procedures. Furthermore, if $f(A)$ and $h(B)$ are tractable, the problem to operate $\exp(A+B)$ to an arbitrary state can be

[2] This idea was briefly reported at the memorial lecture (in Japanese) dedicated to the late Dr. H. Yukawa in Kyoto, March 26, 1982.

solved analytically, as will be discussed explicitly later. A more interesting situation will be *asymptotic separation of procedures*, namely, for example

$$\exp(A+B) \simeq f_{\text{asym}}(A)\, h_{\text{asym}}(B) \tag{6}$$

in some appropriate asymptotic limit, such as the scaling limit.[6,11–15]

Moreover, the concept of *separation of procedures* is very useful as a guiding principle to solving nonlinear problems theoretically. Many typical examples will be given in the succeeding sections.

3. KUBO'S STOCHASTIC LIOUVILLE EQUATION

It will be instructive to discuss here Kubo's method of stochastic Liouville equations[16] from our new point of view, namely on the basis of the concept of separation of procedures.

We consider the following Langevin equation or stochastic differential equation of Stratonovich type[7]:

$$\frac{d}{dt}x = F(x, \eta(t), t) \tag{7}$$

where $\eta(t)$ is a random field.

Kubo[16] introduced the probability density $\rho(x, t)$ for each definite realization of $\eta(t)$. The true distribution function $P(x, t)$ is obtained by taking the average of $\rho(x, t)$ over the distribution of the noise $\eta(t)$. That is, we have

$$P(x, t) = \langle \rho(x, t) \rangle \tag{8}$$

Clearly,[16] $\rho(x, t)$ satisfies the following conservation of the probability:

$$\frac{\partial}{\partial t}\rho(x, t) = -\frac{\partial}{\partial x}[F(x, \eta(t), t)\, \rho(x, t)] \equiv \mathscr{L}(t)\, \rho(x, t) \tag{9}$$

This is the Kubo stochastic Liouville equation. It should be remarked that the two effects of the initial distribution $P(x, 0)$ and the random noise are treated as two separate procedures in the above Kubo method. This has several merits in formal treatments of stochastic processes and their explicit applications.

In fact, the formal solution of (9) is given by

$$\rho(x, t) = \exp_+ \int_0^t \mathscr{L}(s)\, ds \cdot P(x, 0) \tag{10}$$

using the time-ordered exponential defined by[17]

$$\exp_+ \int_0^t \mathscr{L}(s)\, ds = 1 + \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, \mathscr{L}(t_1) \cdots \mathscr{L}(t_n) \tag{11}$$

Thus, our desired probability function $P(x, t)$ is given by

$$P(x, t) = \langle \rho(x, t) \rangle = \left\langle \exp_+ \int_0^t \mathscr{L}(s)\, ds \right\rangle P(x, 0) \tag{12}$$

More explicitly, we consider the following stochastic differential equation:

$$\frac{d}{dt} x(t) = \alpha(x, t) + \beta(x, t)\, \eta(t) \tag{13}$$

with the Gaussian white noise $\eta(t)$ satisfying the relation

$$\langle \eta(t)\, \eta(t') \rangle = 2\varepsilon \delta(t - t') \tag{14}$$

Then we obtain immediately[7,16] the following Fokker–Planck equation

$$\frac{\partial}{\partial t} P(x, t) = \left[-\frac{\partial}{\partial x} \alpha(x, t) + \varepsilon \frac{\partial}{\partial x} \beta(x, t) \frac{\partial}{\partial x} \beta(x, t) \right] P(x, t) \tag{15}$$

Thus, Kubo's stochastic Liouville equation is a typical example of separation of procedures.

4. SEPARATION OF PROCEDURES IN GENERALIZED DIFFUSIONLIKE EQUATIONS

For simplicity we consider first the following linear diffusionlike equation:

$$\frac{\partial}{\partial t} \varphi(\mathbf{r}, t) = \gamma \varphi(\mathbf{r}, t) + D \nabla^2 \varphi(\mathbf{r}, t) + \eta(\mathbf{r}, t) \tag{16}$$

where $\eta(\mathbf{r}, t)$ denotes random noise.

It is easily shown[8] that the above equation is equivalent to the following coupled equations:

$$\varphi(\mathbf{r}, t) = \langle \psi(\mathbf{r}, t) \rangle_\xi \tag{17}$$

and

$$\frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \gamma \psi(\mathbf{r}, t) + \boldsymbol{\xi}(t) \cdot \nabla \psi(\mathbf{r}, t) + \eta(\mathbf{r}, t) \tag{18}$$

where $\xi(t)$ is an auxiliary stochastic variable.[18] It is assumed to be Gaussian and white; namely

$$\langle \xi_i(t)\, \xi_j(t') \rangle = 2D\delta_{ij}\delta(t-t') \tag{19}$$

In fact, the formal solution of (18) is given by

$$\psi(\mathbf{r}, t) = \exp\left[\gamma t + \int_0^t \boldsymbol{\xi}(s)\, ds \cdot \nabla\right] \times \left\{\int_0^t \exp\left[-\gamma s - \int_0^s \boldsymbol{\xi}(u)\, du \cdot \nabla\right] \eta(\mathbf{r}, s)\, ds + \psi(\mathbf{r}, 0)\right\} \tag{20}$$

Then, we have

$$\varphi(\mathbf{r}, t) = \langle \psi(\mathbf{r}, t) \rangle_\xi = e^{\gamma t + tD\nabla^2}\left[\int_0^t e^{-\gamma s - sD\nabla^2} \eta(\mathbf{r}, s)\, ds + \varphi(\mathbf{r}, 0)\right] \tag{21}$$

with $\varphi(\mathbf{r}, 0) = \langle \psi(\mathbf{r}, 0) \rangle_\xi = \psi(\mathbf{r}, 0)$. Equation (21) is clearly the formal solution of (16).

Now we consider the following nonlinear diffusionlike (diffusion-reaction) equation:

$$\frac{\partial}{\partial t}\varphi(\mathbf{r}, t) = \alpha(\varphi(\mathbf{r}, t)) + D\nabla^2\varphi(\mathbf{r}, t) + \eta(\mathbf{r}, t) \tag{22}$$

where $\alpha(x)$ denotes a nonlinear function such as $\alpha(x) = \gamma x - gx^2$ or $\alpha(x) = \gamma x - gx^3$. As is well known, it is extremely difficult to solve (22) analytically. Thus, we consider here the following auxiliary equation:

$$\frac{\partial}{\partial t}\psi(\mathbf{r}, t) = \alpha(\psi(\mathbf{r}, t)) + \boldsymbol{\xi}(t) \cdot \nabla\psi(\mathbf{r}, t) + \eta(\mathbf{r}, t) \tag{23}$$

with the auxiliary Gaussian white noise $\xi(t)$. Rosen[18] showed that the solution of (23) *without random noise* $\eta(t)$ under the initial condition $\psi(\mathbf{r}, 0) = \varphi(\mathbf{r}, 0)$ is a rigorous lower bound on the solution of (22) *without random noise* $\eta(t)$.

The purpose of the present section is to try to solve the auxiliary equation (23) by applying the concept of separation of procedures to the present problem. To approximate (22) by (23) is itself an *approximate separation of procedures.*

Now Eq. (23) is rewritten as

$$\frac{\partial}{\partial t}\tilde{\psi}(\mathbf{r}, t) = \alpha(\tilde{\psi}(\mathbf{r}, t)) + \eta(\mathbf{r} - \mathbf{W}(t), t) \tag{24}$$

by the transformation

$$\tilde{\psi}(\mathbf{r}, t) = \psi(\mathbf{r} - \mathbf{W}(t), t) \tag{25}$$

with

$$\mathbf{W}(t) = \int_0^t \xi(s)\, ds \tag{26}$$

As it is difficult to solve (24) for a general type of nonlinear function $\alpha(x)$, we study here the following typical example:

$$\frac{\partial}{\partial t}\tilde{\psi}(\mathbf{r}, t) = \gamma\tilde{\psi}(\mathbf{r}, t) - g\tilde{\psi}^2(\mathbf{r}, t) + \tilde{\eta}(\mathbf{r}, t) \tag{27}$$

with

$$\tilde{\eta}(\mathbf{r}, t) = \eta(\mathbf{r} - \mathbf{W}(t), t) \tag{28}$$

Without loss of generality, we may put $g = 1$ in (27), and consequently our problem is reduced to solving the following nonlinear Langevin equation:

$$\frac{d}{dt}x = \gamma x - x^2 + \eta(t) \tag{29}$$

By the nonlinear transformation

$$x(t) = \frac{d}{dt}\log u(t) = \frac{du(t)}{dt}\Big/ u(t) \tag{30}$$

we obtain the following linear equation

$$\frac{d^2}{dt^2}u(t) = \gamma\frac{d}{dt}u(t) + \eta(t)\, u(t) \tag{31}$$

If we put $v(t) = du(t)/dt$, then Eq. (29) is transformed into

$$\frac{d}{dt}\mathbf{X}(t) = A(t)\,\mathbf{X}(t) \tag{32}$$

where

$$\mathbf{X}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \qquad \text{and} \qquad A(t) = \begin{pmatrix} 0 & 1 \\ \eta(t) & \gamma \end{pmatrix} \tag{33}$$

Then, the formal exact solution of (32) is given by

$$\mathbf{X}(t)=\left[\exp_{+}\int_{0}^{t}A(s)\,ds\right]\mathbf{X}(0) \tag{34}$$

Consequently we have

$$x(t)=v(t)/u(t)=\mathbf{X}_2(t)/\mathbf{X}_1(t) \tag{35}$$

More explicitly, we have the series expansion

$$u(t)=\sum_{n=0}^{\infty}u_n(t); \qquad u_0(t)=e^{\gamma t}+\left(\frac{\gamma}{x(0)}-1\right) \tag{36}$$

where $u_n(t)$ is given by

$$\begin{aligned} u_n(t)&=\int_0^t dt_1\int_0^{t_1}ds_1\,e^{\gamma(t_1-s_1)}\eta(s_1)\,u_{n-1}(s_1)\\ &=\int_0^t dt_1\int_0^{t_1}ds_1\cdots\int_0^{s_{n-1}}dt_n\int_0^{t_n}ds_n\,\eta(s_1)\cdots\eta(s_n)\\ &\quad\times\exp[\gamma(t_1+\cdots+t_n-s_1-\cdots-s_n)]\,u_0(s_n) \end{aligned} \tag{37}$$

for $n\geqslant 1$. The function $v(t)$ is immediately obtained in the form

$$v(t)=\sum_{n=0}^{\infty}v_n(t); \qquad v_0(t)=\gamma e^{\gamma t} \tag{38}$$

and $v_n(t)=du_n(t)/dt$ with (37).

Thus, we have succeeded in solving formally the nonlinear diffusion-like equation[3]

$$\frac{\partial}{\partial t}\psi(\mathbf{r},t)=\gamma\psi(\mathbf{r},t)-g\psi^2(\mathbf{r},t)+\boldsymbol{\xi}(t)\cdot\nabla\psi(\mathbf{r},t)+\eta(\mathbf{r},t) \tag{39}$$

Then, an approximate solution of the stochastic nonlinear diffusion equation

$$\frac{\partial}{\partial t}\varphi(\mathbf{r},t)=\gamma\varphi(\mathbf{r},t)-g\varphi^2(\mathbf{r},t)+D\nabla^2\varphi(\mathbf{r},t)+\eta(\mathbf{r},t) \tag{40}$$

is given by

$$\varphi(\mathbf{r},t)\simeq\langle\psi(\mathbf{r},t)\rangle_{\xi} \tag{41}$$

[3] See note added in proof of Ref. 8.

This is a typical example of approximate separation of procedures. More explicit applications of the above formal solution will be discussed elsewhere.

5. VAN KAMPEN'S EXPANSION, KUBO'S EXTENSIVITY, AND PRIGOGINE'S ENTROPY PRODUCTION

In this section, we discuss a very important example of separation of procedures in nonequilibrium systems.

For simplicity, we start from the master equation of the form[19–22]

$$\frac{\partial}{\partial t} P(X, t) = -\int W(X \to X', t)\, dX'\, P(X, t) + \int W(X' \to X, t)\, P(X', t)\, dX' \tag{42}$$

where $P(x, t)$ is the probability density of finding X at the value X at the time t and $W(X \to X', t)$ is the transition probability per unit time from X to X'. Following Kubo,[19,20] we assume that the transition probability is proportional to the system size Ω, namely $W(X \to X + r, t) = \Omega w(x, r, t)$ with $x = X/\Omega$, and that $P(X, t) = \varepsilon P(x, t)$ with $\varepsilon = 1/\Omega$. Then, the above master equation (42) is transformed into the following Hamilton–Jacobi form[19–22]

$$\varepsilon \frac{\partial}{\partial t} P(x, t) + \mathscr{H}\left(x, \varepsilon \frac{\partial}{\partial x}, t\right) P(x, t) = 0 \tag{43}$$

where

$$\mathscr{H}(x, p, t) = \int (1 - e^{-rp})\, w(x, r, t)\, dr = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} p^n c_n(x, t) \tag{44}$$

and the nth moment $c_n(x, t)$ is defined by

$$c_n(x, t) = \int r^n w(x, r, t)\, dr \tag{45}$$

Equation (43) with (44) is also called the Kramers–Moyal equation.

The above stochastic process is also expressed by the Langevin equation of the form (7). Van Kampen[23] introduced the following separation of variables:

$$x = y(t) + \varepsilon^{-1/2} \xi(t) \tag{46}$$

Then, the main part $y(t)$ satisfies the following deterministic equation:

$$\frac{d}{dt} y(t) = c_1(y(t), t) \tag{47}$$

and the probability function $\hat{P}(\xi, t)$ of ξ, namely $\hat{P}(\xi, t) = P(y(t) + \varepsilon^{-1/2}\xi(t), t)$, satisfies

$$\frac{\partial}{\partial t}\hat{P}(\xi, t) = -\frac{\partial c_1(y(t), t)}{\partial y(t)}\frac{\partial}{\partial \xi}\xi\hat{P}(\xi, t) + \frac{c_2(y(t), t)}{2}\frac{\partial^2}{\partial \xi^2}\hat{P}(\xi, t) \tag{48}$$

to order $\varepsilon^{1/2}$. Thus, the variable ξ is Gaussian in this approximation, and consequently the variance $\langle \xi^2(t) \rangle \equiv \sigma(t)$ is given by the solution of the equation

$$\frac{d}{dt}\sigma(t) = 2c_1'(y(t), t)\,\sigma(t) + c_2(y(t), t) \tag{49}$$

with $c_1'(x, t) = \partial c_1(x, t)/\partial x$, as is well known.

This is a typical example of separation of procedures, namely the *separation of nonlinearity and fluctuation.*

Kubo proposed the following extensivity ansatz[19,20]:

$$P(X, t) = C \exp[\Omega\varphi(x, t)] \tag{50}$$

for a large system size with $x = X/\Omega$. This is a generalization of the concept of the extensivity of equilibrium statistical thermodynamics to non-equilibrium systems. I have proven[21,22,24] this ansatz under some general conditions. It is easily shown[20-22] that the entropy function $\varphi(x, t)$ satisfies the equation

$$\frac{\partial}{\partial t}\varphi(x, t) + \mathcal{H}\left(x, \frac{\partial}{\partial x}\varphi(x, t), t\right) = 0 \tag{51}$$

If $\mathcal{H}(x, p, t)$ does not depend on the time t, namely $\mathcal{H} = \mathcal{H}(x, p)$, then the equilibrium solution $\varphi_{eq}(x)$ is given by the solution of the equation

$$\mathcal{H}\left(x, \frac{\partial}{\partial x}\varphi_{eq}(x)\right) = \int (1 - e^{-rp_e})\, w(x, r)\, dr = 0 \tag{52}$$

with $p_e \equiv \partial\varphi_{eq}/\partial x$.

If we expand the entropy function $\varphi(x, t)$ as

$$\varphi(x, t) = -\frac{[x - y(t)]^2}{2\sigma(t)} + \cdots \tag{53}$$

then we obtain van Kampen's equations (47) and (49) on $y(t)$ and $\sigma(t)$. Thus, Kubo's extensivity ansatz include van Kampen's limit as a special case. That is, Kubo's extensivity ansatz is also an example of separation of procedures.

It should be remarked here that the function $\Phi(t) = -\varphi_{eq}(y(t))$ plays[22] the role of the Liapunov function or H-function, because

$$\begin{aligned}\frac{d}{dt}\Phi(t) &= -\frac{dy(t)}{dt}\frac{d}{dy(t)}\varphi_{eq}(y(t)) \\ &= -c_1(y(t))\frac{d}{dy(t)}\varphi_{eq}(y(t)) \\ &= \int [1 - rp_e(t) - e^{-rp_e(t)}]\, w(y(t), r)\, dr \leqslant 0 \end{aligned} \tag{54}$$

with $p_e(t) = \partial\varphi_{eq}(y(t))/\partial y(t)$. Here we have used the relation

$$\int (1 - e^{-rp_e(t)})\, w(y(t), r)\, dr = 0 \tag{55}$$

and the inequality that $e^x \geqslant 1 + x$. In fact, it is easily shown in the thermodynamic limit that the H-function defined by

$$H(t) = \frac{1}{\Omega}\int P(x, t) \log[P(x, t)/P_{eq}(x)]\, dx \tag{56}$$

is equal to[22]

$$\Phi(t) = -\varphi_{eq}(y(t)) \tag{57}$$

because we have

$$\langle f(x, t)\rangle_t = \int f(x, t)\, P(x, t)\, dx = f(y(t), t) \tag{58}$$

for an arbitrary function in the thermodynamic limit and because we have $\varphi(y(t), t) = 0$. In the above sense, the function $\varphi(x, t)$ may be called the entropy function, and $\varphi_{eq}(y(t))$ is the relative entropy of the system. Therefore, the function $\sigma_S(t)$ defined by

$$\sigma_S(t) = -\frac{d^2\Phi(t)}{dt^2} = \frac{d^2}{dt^2}\varphi_{eq}(y(t)) \tag{59}$$

plays the role of Prigogine's relative entropy production rate. In fact, we have

$$\sigma_S(t) = \frac{dy(t)}{dt}\frac{\partial}{\partial y(t)}[p_e(t)\, c_1(y(t))] \tag{60}$$

where we have assumed for simplicity that the transition probability w is time-independent. To discuss the sign of this entropy production rate more explicitly, we study the following simple situation in which $c_2(x) = c = \text{const}\ (>0)$ and $c_n(x)$ for $n \geqslant 3$ are all vanishing. This corresponds to the Fokker–Planck equation. Then, we have

$$\mathscr{H}(x, p) = pc_1(x) - \tfrac{1}{2}p^2 c_2(x) \tag{61}$$

Therefore, we get $p_e = 2c_1(x)/c$. Thus, we arrive at

$$\sigma_S(t) = \frac{4}{c} c_1^2(y(t)) \frac{\partial}{\partial y(t)} c_1(y(t)) \tag{62}$$

That is, the sign of the entropy production rate is determined by that of the derivative of the first moment $c_1(y(t))$. Consequently, in our case we arrive finally at the following conclusions: (1) The evolution criterion of the system in stable regions (namely $\partial c_1(x)/\partial x < 0$) is governed by the entropy production minimum principle; i.e., $\sigma_S(t) > 0$, as was discussed by Prigogine[1 3]; but (2) the entropy production rate in unstable regions [namely, for $\partial c_1(x)/\partial x > 0$] is negative; i.e., $\sigma_S(t) < 0$.

The present semiclassical treatment of relaxation and fluctuation is not valid for the relaxation from or near the instability point,[6,11,12] as will be discussed later.

6. SCALING THEORY OF TRANSIENT PHENOMENA, LIE ALGEBRA, AND SEPARATION OF PROCEDURES

As has already been reported,[13,14] the solution of the *linear* Fokker–Planck equation

$$\frac{\partial}{\partial t} P(x, t) = \left(-\frac{\partial}{\partial x} \gamma x + \varepsilon \frac{\partial^2}{\partial x^2} \right) P(x, t) \tag{63}$$

is obtained by using the decomposition formula

$$e^{A+B} = e^A e^{f(\alpha)B}, \qquad f(\alpha) = \frac{1 - e^{-\alpha}}{\alpha} \tag{64}$$

for the two-component Lie algebra (A, B) satisfying the commutation relation $[A, B] = \alpha B$. In the above Fokker–Planck equation (63), we may take[13,14]

$$A = -t\gamma \frac{\partial}{\partial x} x \qquad \text{and} \qquad B = t\varepsilon \frac{\partial^2}{\partial x^2} \tag{65}$$

Then, the formal solution of (63) is given by[13,14]

$$P(x, t) = e^{A+B}P(x, 0) = e^{A}e^{f(\alpha)B}P(x, 0) \tag{66}$$

with $\alpha = 2\gamma t$. The operations e^{A} and $e^{f(\alpha)B}$ can be easily performed analytically,[13,14] as is well known. This is locally extended to the following, more general form[25-27]

$$e^{\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_p A_p} = e^{\beta_1 A_1} e^{\beta_2 A_2} \cdots e^{\beta_p A_p} \tag{67}$$

for the p-component Lie algebra

$$[A_i, A_j] = \sum_{k=1}^{p} c_{ij}^{k} A_k \tag{68}$$

where the renormalized coefficients $\beta_1, \beta_2, \ldots, \beta_p$ are functions of the Lie coefficients $\{c_{ij}^{k}\}$. It is quite simple to obtain $\{\beta_j\}$ when

$$c_{ij}^{k} = 0 \qquad \text{for all} \quad k < \max(i, j) \tag{69}$$

This is equivalent to the solvable Lie algebra, and is also a typical example of separation of procedures. In fact, if we put[15,16]

$$e^{x(\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_p A_p)} = e^{x\alpha_1 A_1} F_2(x) \tag{70}$$

then we find that $F_2(0) = 1$ and that

$$\frac{d}{dx} F_2(x) = \left[e^{-x\alpha_1 A_1} \left(\sum_{j=2}^{p} \alpha_j A_j \right) e^{x\alpha_1 A_1} \right] F_2(x) \tag{71}$$

By the condition (69), Eq. (71) can be written in the form

$$\frac{d}{dx} F_2(x) = [\beta_2(x) A_2 + R_3(x)] F_2(x) \tag{72}$$

where R_3 is a linear combination of the operators $A_3, A_4, \ldots, A_p$. By integration, we obtain

$$F_2(x) = \exp_+ \left\{ \int_0^x [\beta_2(s) A_2 + R_3(s)] \, ds \right\} \tag{73}$$

Repeating the same procedure, we arrive finally at (67) with some explicit expressions for $\{\beta_j\}$.

Now we discuss the general nonlinear case (2) with the nonlinear drift term $\alpha(x) = \gamma x + N(x)$ for $\gamma > 0$, where $N(0) = 0$. The point $x = 0$ is an

unstable point of the system. As has generally been discussed previously,[11,12,15,16] the relaxation of the system from the unstable point $x=0$ is *asymptotically* described by the following decoupled product:

$$P^{(sc)}(x, t) = \exp\left[-t\frac{\partial}{\partial x}\alpha(x) \right] \exp\left[tf(2\gamma t)\varepsilon \frac{\partial^2}{\partial x^2} \right] P(x, 0) \tag{74}$$

This has been called the scaling solution of the relevant system. Thus, the scaling theory of transient phenomena[11,12,15,16] is an example of asymptotic (or approximate) separation of procedures. As is well known, the above scaling solution describes quite well the relaxation, formation of macroscopic order, and fluctuation enhancement. This is complementary to the van Kampen and the Kubo treatments of nonequilibrium systems, as reviewed in Section 5.

7. CAM THEORY OF COOPERATIVE PHENOMENA

It is also of great interest to discuss here the basic idea of the *coherent anomaly method* (CAM) of cooperative phenomena proposed by the present author.[28–33] Our CAM theory is a typical example of separation of procedures, as will be discussed below.

The basic idea of the CAM is to consider many self-consistent (mean-field) approximations systematically for each phenomenon and to extract a common feature inherent to them (namely the *coherent anomaly*), by making an analytic continuation of the degree of approximation. In any cluster-mean-field approximation, the singularity of the relevant response function (such as the susceptibility χ_0) takes the classical form (Curie–Weiss law). The key point of the CAM is that the coefficients or amplitudes of the "classical divergences" obtained in systematic mean-field approximations show "coherent anomalies," namely, they diverge systematically (or coherently) as the degree of approximation increases. By studying these coherent anomalies, we can estimate nonclassical (true) critical exponents.

For example, we consider the susceptibility of ferromagnets. It takes the form

$$\chi_0 \simeq \bar{\chi}(T_c)/\varepsilon; \qquad \varepsilon = (T - T_c)/T_c \tag{75}$$

near the approximate critical point T_c obtained in some approximation. The *mean-field critical coefficient* $\bar{\chi}(T_c)$ depends on each approximation, namely each T_c. The coherent anomaly implies that

$$\bar{\chi}(T_c) \sim \delta(T_c)^{-\psi}; \qquad \delta(T_c) = (T_c - T_c^*)/T_c^* \tag{76}$$

for small $\delta(T_c)$, where ψ denotes the so-called coherent-anomaly exponent and T_c^* the true critical point of the system. The variable $\delta(T_c)$ denotes the degree of approximation. The coherent-anomaly exponent ψ can be estimated to any accuracy from the coherent-anomaly data $\bar{\chi}(T_c)$ as a function of T_c or $\delta(T_c)$, which can be calculated explicitly up to any required digits. This is one of the great merits of the CAM theory. In this sense, the CAM may be called a "precise" theoretical approach to critical phenomena, which corresponds to accurate experimental measurements.

It should be also emphasized here that the divergence of the relevant function (such as the susceptibility) induced by the long-range order due to the bifurcation (namely the Curie–Weiss-type singularity) is separated from the remaining singularity induced by the *intrinsic fluctuation* of the system in the CAM theory. In this sense, the CAM is a typical example of separation of procedures. There have been many applications of the CAM.[28–33]

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